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HYPERREFLECTION GROUPS

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Abstract. We introduce the concept of hyperreflection groups, which are a generalization of Coxeter groups. We prove the Deletion and Exchange Conditions for hyperreflection groups, and we discuss special subgroups and fundamental sectors of hyperreflection groups. In the second half of the paper, we prove that Coxeter groups and graph products of groups are examples of hyperreflection groups.

1. Introduction

This article introduces the concept of hyperreflection groups, which are a fruitful generalization of Coxeter groups (also called reflection groups). A hyperreflection is a kind of multiple reflection. In the case of reflection symmetry on a connected space, the fixed points of the reflection separate the space into two components which are interchanged by the reflection. A hyperreflection is a group action on a connected space whose fixed points separate the space into many components, and for any two components there is a unique group element that maps one to the other. A hyperreflection group is a group that is generated by hyperreflections.

Since hyperreflections need not have order two, they are much more general than reflections. We will prove that graph products of groups are hyperreflection groups. The graph product is a very general construction that includes the weak direct product and the free product as special cases. We will also show that Coxeter groups are hyperreflection groups.

On the other hand, hyperreflection groups are not hopelessly general. They retain many of the properties of Coxeter groups, and many results in the theory of Coxeter groups can be translated to this more general setting.

2. Hypergraphs and Cayley Hypergraphs

In this section, we state the basic definitions concerning hypergraphs, and we define the Cayley hypergraph. The reader who wishes to learn more about hypergraphs is referred to [2]. It should be noted that the term "Cayley hypergraph" is not standard, but it was named by analogy to Cayley graphs, which will be discussed in section 8.

A *hypergraph* is a pair (V, E) where E is a set of nonempty subsets of V. V is called the vertex set and E is called the edge set. A hypergraph differs from a graph insofar as an edge of a hypergraph can contain an arbitrary number of vertices, but an edge of a graph always contains exactly two vertices. A subgraph of a hypergraph (V, E) is a hypergraph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. A hypergraph (V, E) is *disconnected* if there exists a partition of V into two nonempty disjoint subsets such that no edge contains elements from both subsets. A hypergraph is *connected* if it is not disconnected. A *component* of a hypergraph is a maximal connected subgraph that contains at least one vertex. The vertex set of a component will also be called a component. A disconnected hypergraph is the disjoint union of its components.

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A *walk* in (V, E) is an alternating sequence of vertices and edges

$$p = (v_0, e_1, v_1, e_2, \dots, v_n)$$

such that $\{v_{i-1}, v_i\} \subseteq e_i$ for all $1 \leq i \leq n$. A walk may have repeated vertices or edges. We say that the p is a walk of length n from v_0 to v_n. Two vertices u and v belong to the same component if and only if there exists a walk from u to v.

If G is a group and Σ is a collection of nontrivial subgroups of G, then the *Cayley hypergraph* of (G, Σ), denoted $\mathrm{Cay}(G, \Sigma)$, is the hypergraph whose vertex set is G and whose edge set is

$$\{gS \colon (g, S) \in G \times \Sigma\}.$$

The Cayley hypergraph is connected if and only $\bigcup \Sigma$ generates G. G acts on $\mathrm{Cay}(G, \Sigma)$ by left multiplication, and the action is simply transitive on the vertex set.

3. Hyperreflections

In this section, we define hyperreflections and introduce their most elementary properties. We will use hyperreflections in the next section to define hyperreflection groups, which are groups that are generated by hyperreflections.

Let X be a connected space. The term "space" is meant to include graphs, hypergraphs, and topological spaces, but it may refer to any sort of geometric object for which a notion of connectivity can be meaningfully defined (see [12]). If G is a group of automorphisms of X then let $\mathrm{Fix}(G)$ denote the fixed set of G, i.e.

$$\mathrm{Fix}(G) = \{x \in X : \forall g \in G, gx = x\}.$$

A nontrivial subgroup R of $Aut(X)$ is a called a *hyperreflection* if R acts simply transitively on the components of $X \setminus \mathrm{Fix}(R)$. This means that $\mathrm{Fix}(R) \neq X$, and if C_1 and C_2 are two components of $X \setminus \mathrm{Fix}(R)$ then there is a unique $r \in R$ such that $rC_1 = C_2$. The fixed sets of hyperreflections are called *walls*. A hyperreflection of order two is called a *reflection*.

Theorem 3.1. *Let R be a hyperreflection on X and let $1 \neq r_1 \in R$. Then $\mathrm{Fix}(R) = \mathrm{Fix}(r_1) := \{x \in X : r_1 x = x\}$.*

Proof. If $x \in \mathrm{Fix}(R)$ then $rx = x$ for all $r \in R$, so in particular $r_1 x = x$. Now suppose that $x \notin \mathrm{Fix}(R)$, and let C be the component of $X \setminus \mathrm{Fix}(R)$ that contains x. Then $r_1 C$ is also a component of $X \setminus \mathrm{Fix}(R)$. Since R acts simply transitively on the set of components, $r_1 C \neq C$. Therefore $r_1 C \cap C = \emptyset$, hence $r_1 x \neq x$. $\square$

Theorem 3.2. *If R and S are hyperreflections on X, and there exists $t \in R \cap S$ with $t \neq 1$, then $\mathrm{Fix}(R) = \mathrm{Fix}(S)$.*

Proof. By Theorem 3.1, $\mathrm{Fix}(R) = \mathrm{Fix}(t) = \mathrm{Fix}(S)$. $\square$

Theorem 3.3. *If R is a hyperreflection on X and if $\sigma \in Aut(X)$ then $R^\sigma := \sigma R \sigma^{-1}$ is a hyperreflection on X.*

Proof. Note that $\mathrm{Fix}(R^\sigma) = \sigma \mathrm{Fix}(R)$, since $x \in \mathrm{Fix}(R) \iff Rx = x \iff \sigma R \sigma^{-1} \cdot \sigma x = \sigma x$. Also, $X \setminus \mathrm{Fix}(R^\sigma)$ is disconnected since $X \setminus \mathrm{Fix}(R^\sigma) = \sigma(X \setminus \mathrm{Fix}(R))$. Let C_1 and C_2 be components of $X \setminus \mathrm{Fix}(R^\sigma)$. Then $\sigma^{-1} C_1$ and $\sigma^{-1} C_2$ are components of $X \setminus \mathrm{Fix}(R^\sigma)$. So, there is a unique $r \in R$ such that $r \cdot \sigma^{-1} C_1 = \sigma^{-1} C_2$. Let $s = \sigma r \sigma^{-1} \in R^\sigma$. Then $sC_1 = C_2$, and s is unique, because r is unique. Therefore R^σ is a hyperreflection on X. $\square$

Theorem 3.4. *If R and S are hyperreflections on X and if $R \subseteq S$ then $R = S$.*

Proof. $\mathrm{Fix}(R) = \mathrm{Fix}(S)$ by Theorem 3.2. Let $1 \neq s \in S$ and let C be a component of $X \setminus \mathrm{Fix}(S)$. Then C and sC are distinct components of $X \setminus \mathrm{Fix}(S)$, so they are also distinct components of $X \setminus \mathrm{Fix}(R)$. Therefore there is a unique element $r \in R$ such that $rC = sC$. Since S acts freely on the components of $X \setminus \mathrm{Fix}(S)$, it follows that $r = s$. Therefore $S \subseteq R$, hence $R = S$. □

It is not true that distinct hyperreflections are always disjoint. For example, let X be the union of the coordinate axes in the plane, viewed as a topological space. Let R be the subgroup of $Aut(X)$ that is generated by the $90°$ rotation $r(x, y) = (-y, x)$, and let S be the subgroup generated by $s(x, y) = (-2y, x/2)$. Then R and S are both hyperreflections on X, but $r^2 = s^2 \neq 1$.

4. Hyperreflection Systems

Let G be a group, let Σ be a set of nontrivial subgroups of G whose union generates G, and let $X = \mathrm{Cay}(G, \Sigma)$. We say that (G, Σ) is a *hyperreflection system* if the action of each element of Σ by left multiplication on X is a hyperreflection. If (G, Σ) is a hyperreflection system, then the elements of Σ are called *fundamental hyperreflections*.

Note that $(G, \{G\})$ is a hyperreflection system for any nontrivial group G. Such a hyperreflection system is called *trivial*. We say that G is a *hyperreflection group* if there exists a set Σ of *proper* subgroups of G such that (G, Σ) is a hyperreflection system.

We assume for the remainder of the section that (G, Σ) is a hyperreflection system.

Theorem 4.1. *A subgroup T of G fixes the edge gS if and only if $T \subseteq S^g$.*

Proof. T fixes $gS \iff TgS = gS \iff TgSg^{-1} = gSg^{-1} \iff T \subseteq S^g$. □

We will assume for the remainder of this section that (G, Σ) is a hyperreflection system.

Theorem 4.2. *Let $A, B \in \Sigma$ and let $h, k \in G$. Then either $A^h = B^k$ or $A^h \cap B^k = \{1\}$. In particular, if $A \neq B$ then $A \cap B \neq \{1\}$.*

Proof. Suppose that $1 \neq g \in A^h \cap B^k$. Then g fixes the edges hA and kB by Theorem 4.1. By Theorem 3.3, A^h and B^k are hyperreflections. By Theorem 3.2, A^h fixes the edge kB, and B^k fixes the edge hA. Therefore, $A^h \subseteq B^k$ and $B^k \subseteq A^h$ by Theorem 4.1, hence $A^h = B^k$. □

Theorem 4.3. *If T is a subgroup of G, and T is a hyperreflection that fixes the edge gA, then $T = A^g$. In particular, T is a hyperreflection if and only if it is a conjugate of a fundamental hyperreflection.*

Proof. $T \subseteq A^g$ by Theorem 4.1, therefore $T = A^g$ by Theorems 3.3 and 3.4. Since every hyperreflection fixes an edge, and conjugates of hyperreflections are hyperreflections, it follows that T is a hyperreflection if and only if it is conjugate to some $A \in \Sigma$. □

Theorem 4.4. *If $R, S \in \Sigma$, $g, h \in G$, and $\mathrm{Fix}(R^g) = \mathrm{Fix}(S^h)$, then $R^g = S^h$.*

Proof. If $e \in \mathrm{Fix}(R^g) \cap \mathrm{Fix}(S^h)$ then $e = kA$ for some $A \in \Sigma$ and $k \in G$. Therefore $R^g = A^k$ and $S^h = A^k$ by Theorem 4.3, hence $R^g = S^h$. □

5. Words and Reduced Words

Let G be a group and let Σ be a set of nontrivial subgroups of G. A *word* in (G, Σ) of length n is a pair of sequences

$$\boldsymbol{s} \quad = \quad ((s_1, \ldots, s_n),\ (S_1, \ldots, S_n))$$

such that $1 \neq s_i \in S_i \in \Sigma$ for all i. The elements s_i are called letters. If the subgroups in Σ are pairwise disjoint then the S_i are determined uniquely by the s_i. In that case, we will call $(s_1, \ldots, s_n)$ a word, since there is no ambiguity. Recall that if (G, Σ) is a hyperreflection system then the elements of Σ are disjoint by Theorem 4.2.

A word determines certain other important sequences, which we will describe. The word $\boldsymbol{s}$ determines a sequence of partial products $(g_0, \ldots, g_n)$ which may be defined recursively as follows:

$$\begin{aligned} g_0 &= 1_G, \\ g_i &= g_{i-1}s_i \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

The word $\boldsymbol{s}$ is said to *represent* g_n. Two words are *equivalent* if they represent the same group element. A word is *reduced* if there is no shorter word that represents the same element. The length of a group element g is denoted $\ell(g)$, and is defined as the length of a reduced word that represents g.

The word $\boldsymbol{s}$ also determines a *dual word*

$$\boldsymbol{t} = ((t_1, \ldots, t_n),\ (T_1, \ldots, T_n))$$

defined as follows:

$$\begin{aligned} t_i &= g_i g_{i-1}^{-1} = g_{i-1}s_i g_{i-1}^{-1}, \\ T_i &= g_{i-1}S_i g_{i-1}^{-1}. \end{aligned}$$

The reader may verify that

$$g_i = s_1 \cdots s_i = t_i \cdots t_1$$

for all $1 \leq i \leq n$, and that the dual word of $\boldsymbol{t}$ is $\boldsymbol{s}$.

These definitions are best understood in the context of the Cayley hypergraph $\mathrm{Cay}(G, \Sigma)$. The word $\boldsymbol{s}$ corresponds to a walk $(g_0, e_1, g_1, e_2, \ldots, g_n)$ from 1 to g_n, where $e_i = g_{i-1}S_i = g_iS_i$. Since $g_i = g_{i-1}s_i = t_ig_{i-1}$, we have two different ways to describe how we move from one vertex g_{i-1} to the next vertex g_i. We can either multiply by s_i on the right, or multiply by t_i on the left. The subgroup T_i is the stabilizer of the ith edge along the walk.

We will maintain these notations for the remainder of the article, so if a word $\boldsymbol{s}$ is defined, then we consider the sequences (S_i), (g_i), (t_i), and (T_i) to be defined as well.

6. The Deletion and Exchange Conditions

In this section we describe two conditions, called the Deletion and Exchange Conditions, that are satisfied by any hyperreflection system. These conditions illustrate the important role of dual words in the reduction of words in hyperreflection systems. We will assume throughout this section that (G, Σ) is a hyperreflection system.

The first theorem shows that any word in a hyperreflection system can be reduced by successive deletion and replacement of letters. We call this theorem the Deletion Condition, because it generalizes the Deletion Condition for Coxeter groups [4, 8].

Theorem 6.1. *Let $\boldsymbol{s} = (s_1, \cdots, s_n)$ be a word representing $g \in G$. Then the following statements hold.*

(1) *If $t_i = t_j^{-1}$ and $i < j$, then $g = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n$, where the hats indicate that the letters s_i and s_j are to be deleted.*
(2) *If $T_i = T_j$, $t_i \neq t_j^{-1}$, and $i < j$, then there exists $1 \neq \widetilde{s_i} \in S_i$ such that $g = s_1 \cdots \widetilde{s_i} \cdots \widehat{s_j} \cdots s_n$. In other words, we replace s_i with another non-identity element of S_i, and we delete s_j.*

(3) *If* $T_i = T_j$, $t_i \neq t_j^{-1}$, *and* $i < j$, *then there exists* $1 \neq \widetilde{s_j} \in S_j$ *such that* $g = s_1 \cdots \widehat{s_i} \cdots \widetilde{s_j} \cdots s_n$.

(4) $\boldsymbol{s}$ *is reduced if and only if* $T_i \neq T_j$ *for all* $i \neq j$.

Proof. We begin by observing that $s_1 \cdots s_n = t_j t_i s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n$.

$$\begin{aligned} s_1 \cdots s_n &= (s_1 \cdots s_{j-1}) s_j (s_{j+1} \cdots s_n) \\ &= t_j (s_1 \cdots s_{j-1})(s_{j+1} \cdots s_n) \\ &= t_j (s_1 \cdots s_{i-1}) s_i (s_{i+1} \cdots s_{j-1})(s_{j+1} \cdots s_n) \\ &= t_j t_i (s_1 \cdots s_{i-1})(s_{i+1} \cdots s_{j-1})(s_{j+1} \cdots s_n) \\ &= t_j t_i s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n. \end{aligned}$$

If $t_i = t_j^{-1}$ then the last expression is equal to $s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n$, which proves (1).

Suppose that $T_i = T_j$ and $t_i \neq t_j^{-1}$. Then $t_j t_i g_{i-1} S_i \in T_i g_{i-1} S_i = g_{i-1} S_i$, so there exists $1 \neq \widetilde{s_i} \in S_i$ such that $t_j t_i g_{i-1} = g_{i-1} \widetilde{s_i}$. Consequently, $s_1 \cdots s_n = s_1 \cdots \widetilde{s_i} \cdots \widehat{s_j} \cdots s_n$, which proves (2).

One can prove part (3) by applying part (2) to the inverse word $\boldsymbol{s}^{-1} = (s_n^{-1}, \ldots, s_1^{-1})$. If $\boldsymbol{s}$ is reduced then $T_i \neq T_j$ for all $i \neq s$, since otherwise we could reduce the word length by applying (2) or (3). The converse is proved in the next theorem. □

Theorem 6.2. *Let* $\boldsymbol{s} = (s_1, \cdots, s_m)$ *and* $\boldsymbol{s'} = (s'_1, \cdots, s_n)$ *be two words representing* g.

(1) *If* $T_i \neq T_j$ *for all* $i \neq j$ *then* $\{T_1, \cdots, T_m\} \subseteq \{T'_1, \cdots, T'_n\}$

(2) *If* $T_i \neq T_j$ *for all* $i \neq j$ *then* $\boldsymbol{s}$ *is reduced.*

(3) *If* $\boldsymbol{s}$ *and* $\boldsymbol{s'}$ *are both reduced, then* $m = n$ *and* $\{T_1, \cdots, T_m\} = \{T'_1, \cdots, T'_n\}$

Proof. Suppose that $T_i \neq T_j$ for all $i < j$. The word $\boldsymbol{s}$ corresponds to a walk $(g_0, e_1, g_1, \ldots, g_n)$ in $\mathrm{Cay}(G, \Sigma)$, and this walk crosses each wall $\mathrm{Fix}(T_i)$ exactly once. Since these walls separate 1 and g, they must be crossed by every walk from 1 to g. Therefore $T_i \in \{T'_1, \cdots, T'_n\}$ for all i, which implies that $\{T_1, \cdots, T_m\} \subseteq \{T'_1, \cdots, T'_n\}$. This also implies that $m \leq n$, which proves (2).

If $\boldsymbol{s}$ and $\boldsymbol{s'}$ are both reduced, then $T_i \neq T_j$ for all $i < j$, and $T'_i \neq T'_j$ for all $i < j$. Part (1) implies that $\{T_1, \cdots, T_m\} \subseteq \{T'_1, \cdots, T'_n\}$ and $\{T'_1, \cdots, T'_n\} \subseteq \{T_1, \cdots, T_m\}$, hence the two sets are equal. □

Theorem 6.3. *If* $(s_1, \ldots, s_n)$ *and* $(s'_1, \ldots, s'_n)$ *are two reduced words representing* g *then* $\{t_1, \ldots, t_n\} = \{t'_1, \ldots, t'_n\}$.

Proof. Let $i \in \{1, 2, \ldots, n\}$. The walk $(g_0, e_0, g_1, \ldots, g_n)$ crosses each wall $\mathrm{Fix}(T_i)$ exactly once, since $T_i \neq T_j$ for $i \neq j$ by Theorem 6.1. The walk $(g'_0, g'_1, \ldots, g'_n)$ must also cross each wall $\mathrm{Fix}(T_i)$ exactly once, since $T'_i \neq T'_j$ for $i \neq j$. Both walks travel from C_i to $t_i C_i$, so there exists j such that $g'_{j-1} \in C_i$ and $g'_j \in t_i C_i$, which implies that $t'_j = t_i$.

Since i was arbitrary, it follows that $\{t_1, \ldots, t_n\} \subseteq \{t'_1, \ldots, t'_n\}$, therefore $\{t_1, \ldots, t_n\} = \{t'_1, \ldots, t'_n\}$ by symmetry. □

Theorem 6.4. *If* $g \in G$ *and* $S \in \Sigma$ *then the right coset* Sg *has a unique element of minimal length, and the left coset* gS *also has a unique element of minimal length. (cf. Theorem 7.6.)*

Proof. Let w and w' be two distinct elements of minimal length in Sg, and choose a reduced word $(s_1, \ldots, s_n)$ representing w. Then there exists $1 \neq s \in S$ such that $w' = sw = ss_1 \ldots s_n$.

The word $(s, s_1, \ldots, s_n)$ is not reduced, so we can reduce the length via deletion. The deletion must involve the first letter s, since $(s_1, \ldots, s_n)$ is reduced. Therefore, there exists $1 \leq i \leq n$ and $s' \in S$ such that $w' = s's_1 \ldots \widehat{s_i} \ldots s_n$. But $(s')^{-1}w' \in Sg$ and $\ell((s')^{-1}w') < n$, which is a contradiction. This proves that Sg has a unique element of minimal length.

Note that if w is the unique element of minimal length in Sg then w^{-1} is the unique element of minimal length in $g^{-1}S$, and so every left coset of a fundamental hyperreflection has a unique element of minimal length. □

We conclude the section with a theorem that we call the Exchange Condition, which is a generalization of the Exchange Condition for Coxeter groups [4,8].

Theorem 6.5. *Let $\boldsymbol{s} = (s_1, \ldots, s_n)$ be a reduced word for g, and let $1 \neq s_0 \in S_o \in \Sigma$. Then $|\ell(s_0^{-1}g) - \ell(g)| \leq 1$, and the following statements hold.*

1. *If $\ell(s_0^{-1}g) = \ell(g) - 1$ then there exists $1 \leq i \leq n$ such that $g = s_0 s_1 \ldots \widehat{s_i} \ldots s_n$.*
2. *If $\ell(s_0^{-1}g) = \ell(g)$ then there exists $1 \leq i \leq n$ and $\widetilde{s_i} \in S_i \setminus \{1, s_i\}$ such that $g = s_0 s_1 \ldots \widetilde{s_i} \ldots s_n$.*
3. *If $\ell(s_0^{-1}g) = \ell(g) + 1$ then no reduced word for g begins with an element of S_0.*

Proof. $|\ell(s^{-1}g) - \ell(g)| \leq 1$ is a consequence of the triangle inequality for word length.

Suppose that $\ell(s_0^{-1}g) = \ell(g) - 1$. Note that $s_0^{-1}g$ is the unique element of minimal length in $S_0 g$. The word $(s_0^{-1}, s_1, \ldots, s_n)$ for $s_0^{-1}g$ is not reduced, so it can be reduced using the Deletion Condition. Since $(s_1, \ldots, s_n)$ is reduced, the deletion must involve the first letter. Therefore, there exists $1 \leq i \leq n$ and $s \in S_0$ such that $s_0^{-1}g = ss_1 \cdots \widehat{s_i} \cdots s_n$, hence $g = (s_0 s)s_1 \cdots \widehat{s_i} \ldots s_n$.

Let $s' = (s_0 s)^{-1}$. Then $s'g \in S_0 g$ and $\ell(s'g) \leq n - 1$, so $s'g$ has minimal length in $S_0 g$. Therefore $s' = s_0^{-1}$, which implies $s = 1$. Thus, $g = s_1 \cdots \widehat{s_i} \cdots s_n$, which proves part (1).

We prove part (2) by applying the Deletion Condition to $s_0^{-1}g = s_0^{-1}s_1 \cdots s_n$. As before, the deletion must involve the first letter, so $s_0^{-1}g = s_1 \cdots \widetilde{s_i} \cdots s_n$ for some $\widetilde{s_i} \in S_i$. Therefore, $g = s_0 s_1 \cdots \widetilde{s_i} \cdots s_n$. But $\widetilde{s_i} \neq 1$ because $\ell(g) = n$, and $\widetilde{s_i} \neq s_i$ because $s_0 \neq 1$.

If g has a reduced word of length n that begins with $s \in S_0$, then $s_0^{-1}g$ has a word of length n that begins with $s_0^{-1}s$, which proves part (3).

□

7. Special Subgroups and Sectors

Let (G, Σ) be a hyperreflection system. If $\mathcal{A} \subseteq \Sigma$ then let $G_{\mathcal{A}}$ denote the subgroup of G that is generated by $\bigcup \mathcal{A}$. If $\mathcal{A} = \{A\}$ then we will sometimes write G_A instead of $G_{\{A\}}$. We define $G_\emptyset$ to be the identity subgroup. A subgroup of the form $G_{\mathcal{A}}$ is called a *special subgroup*.

Theorem 7.1. *If $\mathcal{A} \subseteq \Sigma$, $R \in \Sigma$, and $G_{\mathcal{A}} \cap R \neq \{1\}$ then $R \in \mathcal{A}$.*

Proof. Let $1 \neq g \in R \subseteq G_{\mathcal{A}}$, and let $(s_1, \ldots, s_n)$ be a word of minimal length such that $1 \neq s_i \in S_i \in \mathcal{A}$ and $g = s_1 \cdots s_n$.

We claim that $(s_1, \ldots, s_n)$ is reduced. Suppose not; then by Theorem 6.1 we can reduce the word to a shorter word by successive deletions. But each letter in the shorter word also belongs to $\bigcup \mathcal{A}$, and this contradicts minimality. The shortest possible word for g has length 1, since $g \in R \in \Sigma$. Therefore, $g \in R \cap S_1$, so $R = S_1 \in \mathcal{A}$ by Theorem 4.2. □

Theorem 7.2. *If $(s_1, \ldots, s_n)$ and $(s'_1, \ldots, s'_n)$ are two reduced words for g then $\{S_1, \ldots, S_n\} = \{S'_1, \ldots, S'_n\}$.*

Proof. Let $\mathcal{A} = \{S_1, \ldots, S_n\}$ and $\mathcal{B} = \{S'_1, \ldots, S'_n\}$. Then $t_i \in G_{\mathcal{A}}$ for all i, so $t'_i \in G_{\mathcal{A}}$ for all i by Theorem 6.3. Since $w'_i = t'_i \cdots t'_1$ for all i, it follows that $w'_i \in G_{\mathcal{A}}$ for all i. But $w'_i = w'_{i-1}s'_i$, thus $s'_i \in G_{\mathcal{A}}$ for all i. Therefore, $S'_i \cap G_{\mathcal{A}} \neq \{1\}$ for all i, hence $S'_i \in \mathcal{A}$ for all i by Theorem 7.1. This shows that $\mathcal{B} \subseteq \mathcal{A}$. By symmetry, $\mathcal{A} \subseteq \mathcal{B}$, therefore $\mathcal{A} = \mathcal{B}$. □

Theorem 7.3. *If $(s_1, \cdots, s_m)$ is a reduced word for g, and $(s'_1, \cdots, s'_n)$ is any other word representing g, not necessarily reduced, then $\{S_1, \cdots, S_m\} \subseteq \{S'_1, \cdots, S'_n\}$.*

Proof. If $(s'_1, \cdots, s'_n)$ is not reduced then it may be transformed to an equivalent reduced word $(s''_1, \ldots, s''_p)$ by successive deletions (Theorem 6.1). It is clear that $\{S''_1, \cdots, S''_p\} \subseteq \{S'_1, \cdots, S'_n\}$. By Theorem 7.2, $\{S''_1, \cdots, S''_p\} = \{S_1, \cdots, S_m\}$. Therefore, $\{S_1, \cdots, S_m\} \subseteq \{S'_1, \cdots, S'_n\}$. □

Theorem 7.4. *If $\mathcal{A} \subseteq \Sigma$ then $(G_{\mathcal{A}}, \mathcal{A})$ is a hyperreflection system.*

Proof. Let $X_0 = \mathrm{Cay}(G_{\mathcal{A}}, \mathcal{A})$ and $X = \mathrm{Cay}(G, \Sigma)$. Note that X_0 is a connected subgraph of X. Let $A \in \mathcal{A}$ be given. Let C be the identity component of $X \setminus \mathrm{Fix}(A)$ and let $C_0 = C \cap X_0$.

Now $X = \mathrm{Fix}(A) \cup \bigcup_{a \in A} aC$, so $X_0 = (\mathrm{Fix}(A) \cap X_0) \cup \bigcup_{a \in A}(aC \cap X_0) = (\mathrm{Fix}(A) \cap X_0) \cup \bigcup_{a \in A} aC_0$. Moreover, $\mathrm{Fix}(A) \cap X_0$ is identical to the fixed set of the action of A on X_0, since X_0 is invariant under A. Since the unions in the expression of X are disjoint, the unions in the expression of X_1 are also disjoint. So it only remains to show that C_0 is connected.

Let g be any vertex of C_0. This implies that $g \in G_{\mathcal{A}}$, and there exists a word $(s_1, \ldots, s_m)$ representing g whose path does not cross $\mathrm{Fix}(A)$, i.e. $T_i \neq A$ for all $1 \leq i \leq n$. Let $(s'_1, \ldots, s'_n)$ be a reduced word representing g. Then $S'_i \in \mathcal{A}$ for all i by Theorem 7.3, and $T'_i \neq A$ for all i by Theorem 6.2. Therefore, the walk $(g'_0, e'_1, g'_1, \ldots, g'_n)$ lies in C_0, hence C_0 is connected. □

Theorem 7.5. *If $\mathcal{A}, \mathcal{B} \subseteq \Sigma$ then $G_{\mathcal{A} \cap \mathcal{B}} = G_{\mathcal{A}} \cap G_{\mathcal{B}}$.*

Proof. Since $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B}$ it follows that $G_{\mathcal{A} \cap \mathcal{B}} \subseteq G_{\mathcal{A}}$ and $G_{\mathcal{A} \cap \mathcal{B}} \subseteq G_{\mathcal{B}}$. Therefore, $G_{\mathcal{A} \cap \mathcal{B}} \subseteq G_{\mathcal{A}} \cap G_{\mathcal{B}}$.

To prove the reverse inclusion, let $g \in G_{\mathcal{A}} \cap G_{\mathcal{B}}$, and let $(s_1, \ldots, s_n)$ be a reduced word for g. Then $S_i \in \mathcal{A}$ for all i by Theorem 7.3, and $S_i \in \mathcal{B}$ for the same reason. Therefore, $S_i \in \mathcal{A} \cap \mathcal{B}$ for all i, hence $g \in G_{\mathcal{A} \cap \mathcal{B}}$. □

Theorem 7.6. (cf. [4, p. 47]). *Suppose that $\mathcal{A}, \mathcal{B} \subseteq \Sigma$ and that w is an element of minimum length in the double coset $G_{\mathcal{A}} w G_{\mathcal{B}}$. Then any element w' in this double coset can be written in the form $w' = awb$ where $a \in G_{\mathcal{A}}$, $b \in G_{\mathcal{B}}$, and $\ell(w') = \ell(a) + \ell(w) + \ell(b)$. In particular, the double coset has a unique element of minimum length.*

Proof. Choose $a \in G_{\mathcal{A}}$ and $b \in G_{\mathcal{B}}$ such that $w' = awb$ and $\ell(a) + \ell(b)$ is as small as possible. Choose reduced words $\boldsymbol{r}$, $\boldsymbol{s}$, and $\boldsymbol{t}$ for a, w, and b, respectively. If the concatenation $\boldsymbol{rst}$ is not reduced, then we can produce an equivalent word of shorter length by deleting two letters, or by deleting one letter and replacing another letter. The two letters cannot occur in the same subword ($\boldsymbol{r}$, $\boldsymbol{s}$, or $\boldsymbol{t}$), because these subwords are reduced. There are three cases to consider:

(1) Delete a letter from $\boldsymbol{s}$, and delete or replace a letter from $\boldsymbol{r}$.
(2) Delete a letter from $\boldsymbol{s}$, and delete or replace a letter from $\boldsymbol{t}$.
(3) Delete a letter from $\boldsymbol{r}$, and delete or replace a letter from $\boldsymbol{t}$.

Note that deleting or replacing a letter from $\boldsymbol{r}$ or $\boldsymbol{t}$ yields another element of the same special subgroup. Thus, the first two cases are impossible as they would yield an element of the double coset that is shorter than w, and the third case is impossible because it contradicts the minimality of $\ell(a) + \ell(b)$. Therefore, $\boldsymbol{rst}$ is reduced, hence

$$\ell(w') = \ell(awb) = \ell(a) + \ell(w) + \ell(b).$$

If w and w' both have minimal length in the double coset, then $\ell(a) = \ell(b) = 0$, thus $w' = w$. □

Given a subset $\mathcal{A}$ of Σ, let $G^{\mathcal{A}}$ be the set of all $g \in G$ such that g is the unique element of minimal length in the coset $gG_{\mathcal{A}}$. $G^{\mathcal{A}}$ is called the *fundamental $\mathcal{A}$-sector*.

Theorem 7.7. *If $g \in G$ and $\mathcal{A} \subseteq \Sigma$ then there exist unique elements $h \in G^{\mathcal{A}}$ and $k \in G_{\mathcal{A}}$ such that $g = hk$.*

Proof. Let h be the unique minimal element $gG_{\mathcal{A}}$. Then $h \in G^{\mathcal{A}}$ because $gG_{\mathcal{A}} = hG_{\mathcal{A}}$. Since $g \in hG_{\mathcal{A}}$, there exists a unique $k \in G_{\mathcal{A}}$ such that $h = hk$. □

8. Coxeter Groups as Hyperreflection Systems

A Coxeter group is a group having a presentation of the form

$$W = \langle S \mid (st)^{m(s,t)} = 1 \ (s, t \in S)\rangle$$

where S is a finite set of generators of W, $m(s,s) = 1$ for all $s \in S$, and $m(s,t) = m(t,s) \in \{2, 3, 4, ..., \infty\}$ for all $s, t \in S$ with $s \neq t$. If $m(s,t) = \infty$ then the corresponding relation is omitted. It can be shown that $m(s,t)$ is the order of st in W [8, p. 110]. The pair (W, S) is called a Coxeter system, and S is a set of Coxeter generators for W.

Coxeter groups appear in nature as the symmetry groups of regular polytopes, and they are important in the theory of Lie algebras, where they arise as subgroups of the isometry groups of root systems [7]. The reader who wishes to learn more about Coxeter groups is referred to [4] or [8].

There is an alternative characterization of Coxeter groups, due to Michael Davis. He proves in [4, Thm 3.3.4] that (W, S) is a Coxeter system if and only if the Cayley graph of (W, S) is a reflection system. The Cayley graph $\mathrm{Cay}(W, S)$ is defined as the graph whose vertex set is W and whose edge set is $\{\{w, ws\} : w \in W, s \in S\}$. Observe that $\{w, ws\} = w\langle s\rangle$, so we may identify $\mathrm{Cay}(W, S)$ with $\mathrm{Cay}(W, \Sigma)$ where $\Sigma = \{\langle s\rangle : s \in S\}$. The definition of a reflection system is rather involved, but in the case of a Cayley graph it reduces to the assertion that each element of S acts by reflection on the Cayley graph. Therefore, if (W, S) is a Coxeter system then (W, Σ) is a hyperreflection system.

9. Graph Products of Groups

The next two sections incorporate material from a preprint by the present author [13].

Given a graph with nontrivial groups as vertices, a group is formed by taking the free product of the vertex groups, with added relations implying that elements of adjacent groups commute. This group is said to be the *graph product* of the vertex groups. If the graph is discrete then the graph product is the free product of the vertex groups; while if the graph is complete then the graph product is the weak direct product of the vertex groups. See [9] for the definitions of the free product and weak direct product of groups. Graph products were first defined in Elizabeth Green's Ph.D. thesis [5], and have been studied by many other authors [3, 6, 11]. In this section, we will characterize graph products of groups by a universal mapping property, and we will present two constructions of the graph product.

Let $\Gamma = (V, E)$ be a graph, and let $\{G_v\}_{v \in V}$ be a collection of groups which is indexed by the vertex set of Γ. We say that $(\Gamma, \{G_v\}_{v \in V})$ is a *graph of groups.* (This differs from the usual definition, which has vertex groups and edge groups, together with monomorphisms from the edge groups to the vertex groups. See [1].)

A *graph product* of a graph of groups consists of a group G and a collection of homomorphisms $e_v\colon G_v \to G$ such that the following conditions hold.

(1) If $\{u, v\} \in E$ then $[e_u(x), e_v(y)] = 1$ for all $x \in G_u$, $y \in G_v$.
(2) If $h_v\colon G_v \to H$ is a collection of homomorphisms such that $[h_u(x), h_v(y)] = 1$ whenever $\{u, v\} \in E$, then there is a unique homomorphism $\phi\colon G \to H$ such that $\phi \circ e_v = h_v$ for all $v \in V$.

Theorem 9.1. *The homomorphisms e_v in the definition of graph product are injective, and G is generated by the union of the images of the e_v.*

Proof. Let $v \in V$, and define a homomorphism $h_u\colon G_u \to G_v$ for each $u \in V$ as follows. If $u = v$ then $h_u(g) = g$ for all $g \in G_u$, otherwise $h_u(g) = 1$ for all $g \in G_u$. Since $[h_u(x), h_w(y)] = 1$ for all $u \neq w$, there is a unique homomorphism $\phi\colon G \to G_v$ such that $\phi \circ e_u = h_u$ for all u. But h_v is injective, hence e_v is also injective. (Note that it also follows that ϕ is surjective.)

As for the other assertion, let G_0 denote the subgroup of G which is generated by the union of the images of the e_v. It is required to prove that $G_0 = G$. Let $j\colon G_0 \to G$ be the inclusion homomorphism, and let h_v denote the co-restriction of e_v to G_0. That is, $j \circ h_v = e_v$ for all $v \in V$. By the universal mapping property of graph products, there is a unique homomorphism $\phi\colon G \to G_0$ such that $\phi \circ e_v = h_v$ for all $v \in V$. Therefore $(j \circ \phi) \circ e_v = j \circ h_v = e_v$ for all $v \in V$. But $id_G \circ e_v = e_v$ so it follows from the universal mapping property that $j \circ \phi = id_G$. Therefore $G_0 = G$ as claimed. □

Theorem 9.2. *If (G, e_v) and (H, f_v) are two graph products of $(\Gamma, \{G_v\}_{v \in V})$ then there exists an isomorphism $\phi\colon G \to H$ such that $\phi \circ e_v = f_v$. In other words, the graph product of a graph of groups is unique up to isomorphism.*

Proof. By the definition of graph product there is a unique homomorphism $\phi\colon G \to H$ so that $f_v = \phi \circ e_v$, and there is a unique homomorphism $\psi\colon H \to G$ so that $e_v = \psi \circ f_v$. Therefore $(\psi \circ \phi) \circ e_v = e_v$ and $(\phi \circ \psi) \circ f_v = f_v$. On the other hand, $id_G \circ e_v = e_v$ and $id_H \circ f_v = f_v$, so by the uniqueness property we have $\psi \circ \phi = id_G$ and $\phi \circ \psi = id_H$. Therefore ϕ is an isomorphism from G to H. □

If each G_v is a subgroup of G, and each e_v is an inclusion homomorphism, then we say that G is the *internal graph product* of the G_v. In this case we suppress mention of the e_v, and say that G is the graph product of the subgroups G_v. In general, if (G, e_v) is the graph product of $(\Gamma, \{G_v\}_{v \in V})$, then G is the internal graph product of the subgroups $e_v(G_v)$.

It remains to prove that graph products exist. We will give two different constructions.

Let F denote the free product of the G_v. By definition, there exist monomorphisms $\iota_v\colon G_v \to F$ such that the following condition is satisfied: for any family of homomorphisms $h_v\colon G_v \to H$ there is a unique homomorphism $\psi\colon F \to H$ such that $\psi \circ \iota_v = h_v$ for all $v \in V$.

Let N denote the normal closure in F of the set of all commutators $[\iota_u(x), \iota_v(y)]$ where u and v are adjacent vertices, $x \in G_u$ and $y \in G_v$. Let $\pi\colon F \to F/N$ be the quotient homomorphism and let $e_v = \pi \circ \iota_v$ for all $v \in V$.

Theorem 9.3. *With the above definitions, $(F/N, e_v)$ is the graph product of $(\Gamma, \{G_v\}_{v \in V})$.*

Proof. Let H be a group, and let $h_v\colon G_v \to H$ be a collection of homomorphisms so that $[h_u(x), h_v(y)] = 1$ whenever $\{u, v\} \in E$, $u \in G_u$ and $v \in G_v$. By the definition of free product there is a unique homomorphism $\psi\colon F \to H$ such that $\psi \circ \iota_v = h_v$ for all $v \in V$.

If $\{u, v\} \in E$, $x \in G_u$ and $y \in G_v$ then $\psi([\iota_u(x), \iota_v(y)]) = [h_u(x), h_v(y)] = 1$, so $N \subseteq \ker(\psi)$. Therefore there is an induced homomorphism $\phi\colon F/N \to H$ such that $\phi = \psi \circ \pi$. It follows that $\phi \circ e_v = h_v$ for all $v \in V$, as $\phi \circ e_v = \phi \circ \pi \circ \iota_v = \psi \circ \iota_v = h_v$.

It remains to show that ϕ is unique. To that end, let $\phi' \colon F/N \to H$ be a homomorphism such that $\phi' \circ e_v = h_v$ for all $v \in V$, and let $\psi' = \phi' \circ \pi$. Then $\psi' \circ \iota_v = \phi' \circ \pi \circ \iota_v = \phi' \circ e_v = h_v$ for all $v \in V$. Therefore $\psi' = \psi$, by uniqueness of ψ. Since π is surjective and $\phi \circ \pi = \phi' \circ \pi$, it follows that $\phi = \phi'$.

Therefore $(F/N, e_v)$ is the graph product of $(\Gamma, \{G_v\}_{v\in V})$. □

We describe another construction of the graph product. Let X denote the set of all finite sequences $(g_1, \ldots, g_n)$ where $1 \neq g_i \in G_{v_i}$ and $v_i \in V$ for all i from 1 to n. We assume that the G_v are pairwise disjoint except for a common identity element 1. A sequence of this type is called a *word*, and each entry is a *syllable*. The *length* of a word is the number of syllables. We admit the empty word $\lambda = (\ )$, which has length 0.

Given two words $w = (g_1, \ldots g_n)$ and $x = (h_1, \ldots, h_m)$, the *product* wx is defined by concatenation: $wx = (g_1, \ldots, g_n, h_1, \ldots, h_m)$. This product is associative and it has an identity element λ, so it gives X the structure of a monoid. The *inverse* of w is defined as $w^{-1} = (g_n^{-1}, \ldots, g_1^{-1})$, although it must be noted that w^{-1} is not the multiplicative inverse of w in X. In fact λ is the only element of X which has a multiplicative inverse.

We define non-negative integer powers by the following recursive definition.

$$w^n = \begin{cases} \lambda & \text{if } n = 0, \\ ww^{n-1} & \text{if } n \geq 1. \end{cases}$$

This is extended to negative integer powers by defining $w^{-n} = (w^{-1})^n$ for $n \geq 2$.

Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, where

$$\begin{aligned} \mathcal{R}_1 &= \bigcup_{v\in V} \left\{ \left((g, g^{-1}),\, \lambda \right) \ : \ g \in G_v \right\}, \\ \mathcal{R}_2 &= \bigcup_{v\in V} \left\{ \left((g, h),\, (gh) \right) \ : \ g, h \in G_v,\ gh \neq 1 \right\}, \quad \text{and} \\ \mathcal{R}_3 &= \bigcup_{\{u,v\}\in E} \left\{ \left((g, h),\, (h, g) \right) \ : \ g \in G_u,\ h \in G_v \right\}. \end{aligned}$$

We say that two words r and s are *elementarily equivalent*, denoted $r \approx s$, if there exist words w, x, y, z such that $r = wxz$, $s = wyz$, and either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. Furthermore, r and s are said to be *equivalent*, denoted $r \sim s$, if there exists a finite sequences of words $w_0, w_1, \ldots, w_n$ such that $w_0 = r$, $w_n = s$, and $w_{i-1} \approx w_i$ for all i from 1 to n.

In more intuitive terms, two words are equivalent if the first word can be transformed to the second word by means of the following moves and their inverses.

(1) If a syllable g is followed by its inverse g^{-1}, then delete both syllables.
(2) If two successive syllables g and h belong to the same vertex group, and if $gh \neq 1$, then replace the two syllables with the single syllable gh.
(3) If two successive syllables g and h belong to adjacent vertex groups, then swap g and h.

Sometimes we will allow words to contain the identity element 1 as a syllable. In that case we add another rule stating that 1's can be deleted.

It is clear that the relation defined above is an equivalence relation. Moreover, it preserves multiplication[1] — if $w \sim x$ and $y \sim z$ then $wy \sim xz$. Let Ω be the set of equivalence classes of X. Then Ω inherits from X the structure of a monoid. In fact Ω is a group, since $ww^{-1} \sim \lambda$ and $w^{-1}w \sim \lambda$ for all $w \in X$. We will write 1 for the identity element $[\lambda]$ of Ω.

[1] In other words, $\sim$ is a congruence.

Define $\pi\colon X \to \Omega$ by $\pi(x) = [x]$. There are natural homomorphisms $e_v\colon G_v \to \Omega$ defined by $e_v(g) = [(g)]$ for $g \neq 1$ and $e_v(1) = 1$.

Theorem 9.4. *With the above definitions, (Ω, e_v) is the graph product of $(\Gamma, \{G_v\}_{v\in V})$.*

Proof. Let $\{u, v\} \in E$, $1 \neq x \in G_u$, and $1 \neq y \in G_v$. Then $e_v(xy) = e_v(yx)$, since $(x, y) \approx (y, x)$. Therefore $e_v([x, y]) = 1$, and the first condition in the definition of graph product is verified.

Let $h_v\colon G_v \to H$ be any collection of homomorphisms with the property that $[h_u(x), h_v(y)] = 1$ for all $x \in G_u$, $y \in G_v$ when $\{u, v\} \in E$.

Define $\psi\colon X \to H$ as follows. If $w = (g_1, \ldots, g_n)$ and $g_i \in G_{v_i}$ for all i, then let $\psi(w) = h_{v_1}(g_1) \cdots h_{v_n}(g_n)$. If $w = \lambda$ then let $\psi(w) = 1$.

It is clear that ψ is a monoid homomorphism and that it respects equivalence. Therefore there is a group homomorphism $\phi\colon \Omega \to H$ such that $\psi = \phi \circ \pi$.

Now if $1 \neq g \in G_v$ then $\phi \circ e_v(g) = \phi(\ [(g)]\ ) = \psi(\ (g)\ ) = h_v(g)$. Therefore $\phi \circ e_v = h_v$ for all $v \in V$. It remains to show that ϕ is unique. To that end, let $\phi'\colon \Omega \to H$ be a homomorphism such that $\phi \circ e_v = h_v$. If $1 \neq g \in G_v$ then $\phi'(\ [(g)]\ ) = \phi' \circ e_v(g) = h_v(g) = \phi \circ e_v(g) = \phi(\ [(g)]\ )$. But Ω is generated by elements of the form $[(g)]$. Therefore $\phi = \phi'$, and the proof is complete. $\square$

10. Normal Forms for Elements of Graph Products

Let $(\Gamma, \{G_v\}_{v\in V})$ be a graph of groups, with graph product G. We will realize G as the group of equivalence classes of words from Theorem 9.4. Let X be the set of words used in this construction.

Choose an arbitrary linear ordering $\prec$ of V. Let $w = (g_1, \ldots, g_n)$ be a word, where $1 \neq g_i \in G_{v_i}$ and $v_i \in V$ for all i. We say that w is *reduced* if it is not equivalent to any shorter word. We say that w is *normal* if it satisfies the following conditions:

(1) $v_i \neq v_{i+1}$ for all i between 1 and $n - 1$, and
(2) if $\{v_i, v_{i+1}\} \in E$ then $v_i \prec v_{i+1}$.

We also consider λ to be a normal word.

Theorem 10.1. *Every element of G is represented by exactly one normal word. The unique normal word representing g is called the* normal form *of g.*

Proof. The following argument is modeled on the proof by van der Waerden of the normal form theorem for free products [10, 14]. The theorem was first proved by Green [5].

For each $v \in V$ we define $\mu_v\colon G_v \times X \to X$ by the following recursive algorithm. Let $g \in G_v$ and let $x \in R$.

(1) If $g = 1$ then $\mu_v(g, x) = x$.
(2) If $g \neq 1$ and $x = \lambda$ then $\mu_v(g, x) = (g)$.
(3) Suppose that $g \neq 1$ and $x \neq \lambda$. Let g_1 be the first syllable of x. Select $v_1 \in V$ such that $g_1 \in G_{v_1}$, and select $y \in X$ such that $x = (g_1)\ y$.
    (a) If $v = v_1$ and $gg_1 = 1$ then $\mu_v(g, x) = y$.
    (b) If $v = v_1$ and $gg_1 \neq 1$ then $\mu_v(g, x) = (gg_1)\ y$.
    (c) If $v_1 \prec v$ and $\{v, v_1\} \in E$ then $\mu_v(g, x) = (g_1)\ \mu_v(g, y)$.
    (d) Otherwise $\mu_v(g, x) = (g)\ x$.

Let R denote the set of normal words of X. We claim that if $x \in R$, $v \in V$ and $g \in G_v$ then $\mu_v(g, x) \in R$. The proof is by induction on word length. Suppose that $x \in R$, and that $\mu_v(g, y)$ is normal for every reduced word y which is shorter than x. We need to show that

$\mu_v(g,x) \in R$. This is done by checking each of the six cases in the recursive definition. The verification of these cases is left to the reader.

A similar case-by-case analysis shows that if $g, h \in G_v$ and $x \in R$ then $\mu_v(g, \mu_v(h,x)) = \mu_v(gh, \mu_v(x))$. Since $\mu_v(g, \mu_v(g^{-1}, x)) = x$, it follows that $\mu_v(g, \cdot)$ is a permutation of R for each $g \in G$. Therefore there is a homomorphism $h_v \colon G_v \to \mathit{Perm}(R)$ defined by $h_v(g) = \mu_v(g, \cdot)$.

It is a routine matter to verify that $[h_u(g), h_v(k)] = 1$ whenever $\{u, v\} \in E$. Therefore there exists a homomorphism $\phi \colon G \to \mathit{Perm}(R)$ such that $\phi(g) = \mu_v(g, \cdot)$ when $g \in G_v$.

This homomorphism allows us to compute for any word x an equivalent normal word w. Let $x = (x_1, \dots, x_n)$ and let $w = \phi([x])(\lambda)$. Then w is a normal word, and w is equivalent to x. On the other hand, $w = \phi([w])(\lambda)$, so w is the only normal word which is equivalent to x. For if w' is another normal word equivalent to x, then $w' = \phi([w'])(\lambda) = \phi([w])(\lambda) = w$. □

Theorem 10.2. *Every normal word is reduced.*

Proof. Let x be a normal word, and let y be a reduced word which is equivalent to x. If y contains two successive syllables g_i and g_{i+1} such that $\{v_i, v_{i+1}\} \in E$ and $v_{i+1} \prec v_i$, then swap these syllables. Repeat this until no more swaps are possible. This must terminate because no pair of syllables can be swapped more than once. Let z be the word which results. Now z cannot have two successive syllables belonging to the same vertex group, else y would not have minimal length. Therefore z is normal, hence $z = x$ by the previous theorem. Since y and z have the same length, it follows that x is reduced. □

Corollary 10.3. *A reduced word can be transformed into an equivalent normal word by swapping syllables belonging to adjacent vertex groups.* □

11. Graph Products as Hyperreflection Systems

Let (V, E) be a graph, and suppose that G is the internal graph product of a collection of subgroups $\{G_v\}_{v \in V}$. Also suppose that (G_v, Σ_v) is a hyperreflection system for each $v \in V$. The main objective of this section is to prove that (G, Σ) is a hyperreflection system, where $\Sigma = \bigcup_{v \in V} \Sigma_v$.

Define a weight function wt on G as follows. For each $v \in V$ let ℓ_v be the length function associated to (G_v, Σ_v), and let $\ell = \bigcup_{v \in V} \ell_v$. If $(g_1, \dots, g_n)$ is the normal form for g, and if $g_i \in G_{v_i}$ for each i, then define $\mathrm{wt}(g) = \sum_{i=1}^{n} \ell(g_i)$.

Let $v \in V$ and $S \in \Sigma_v$ be given. Choose a linear ordering $\prec$ on V such that v is minimal with respect to $\prec$. This linear ordering determines a normal form for the elements of G.

Theorem 11.1. *If $g \in G$ then Sg has a unique element of minimum weight, and g is the minimum weight element of Sg if and only if $g_1 \in (G_v)^S$ or $g_1 \notin G_v$, where g_1 is the first syllable of the normal form for g.*

Proof. Let $(g_1, \dots, g_n)$ be the normal form for g. We consider three cases.

(1) Suppose that $g_1 \notin G_v$. If $1 \neq s \in S$ then $(s, g_1, \dots, g_n)$ is the normal form for sg. Therefore, $\mathrm{wt}(sg) = \mathrm{wt}(g) + 1$ for all $1 \neq s \in S$, so g is the unique element of minimum weight in Sg.
(2) Suppose that $g_1 \in G_v$ and $g_1 \notin S$. If $s \in S$, then $(sg_1, \dots, g_n)$ is the normal form for sg. Therefore, $\mathrm{wt}(sg)$ is minimized when $\ell(sg_1)$ is minimized. But the coset Sg_1 of G_v has a unique element of minimum length, hence Sg has a unique element of minimum weight. Furthermore, g is the minimum weight element of Sg if and only if g_1 is the minimum length element of Sg_1, which occurs precisely when $g_1 \in (G_v)^S$.

(3) Suppose that $g_1 \in S$. If $s = g_1^{-1}$ then $(g_2, \ldots, g_n)$ is the normal form for sg, otherwise $(sg_1, g_2, \ldots, g_n)$ is the normal form for sg. The weight is uniquely minimized when $s = g_1^{-1}$, hence Sg has a unique element of minimum weight. Note that g cannot be the minimum weight element of Sg in this case, since $\mathrm{wt}(g_1^{-1}g) = \mathrm{wt}(g) - 1$.

□

Theorem 11.2. (G, Σ) *as defined above is a hyperreflection system.*

Proof. Let C be the set of all elements $g \in G$ such that g is the unique element of minimum weight in Sg. Since every coset Sg has a unique element of minimum weight, it follows that G is the disjoint union of sC for $s \in S$.

Let $g \in C$ and let $(g_1, \ldots, g_n)$ be the normal form for g. Choose a reduced word $\boldsymbol{s_i}$ for each g_i and let $\boldsymbol{s} = (s_1, \ldots, s_N)$ be the concatenation of the $\boldsymbol{s_i}$. Then $\boldsymbol{s}$ determines a walk π from 1 to g in $\mathrm{Cay}(G, \Sigma)$.

Suppose that π crosses from C to sC for some $s \in S \setminus \{1\}$, i.e. there exists k such that

$$s(s_1 \ldots s_{k-1}) = (s_1 \cdots s_k).$$

Then $\boldsymbol{s}$ is equivalent to a new word

$$\boldsymbol{s'} = (s, s_1, \ldots, \widehat{s_k}, \ldots, s_N).$$

Let i be the index such that s_k lies in the subword $\boldsymbol{s_i}$, which represents the syllable g_i. If $g_i = s_k$ then g can be represented by $(s, g_1, \ldots, \widehat{g_i}, \ldots, g_n)$. Since this word has n syllables, it is reduced; so its normal form is obtained by swapping adjacent syllables. The first syllable s cannot be swapped, because v is the first vertex in the chosen linear ordering of V. Therefore, s must be the first syllable in the normal form of g, which contradicts the assumption that $g \in C$.

If $g_i \neq s_k$ then g can be represented by $(s, g_1, \ldots, g_i', \ldots, g_n)$, where g_i' is obtained by deleting s_k from $\boldsymbol{s_i}$. Since this word has $n+1$ syllables, it is not normal. However, $(g_1, \ldots, g_i', \ldots, g_n)$ is normal, and this implies (by the normal form algorithm) that $g_1 \in S$ and that

$$(sg_1, g_2, \ldots, g_i', \ldots, g_n)$$

is the normal form for g. Therefore $sg_1 = g_1$ by the uniqueness of normal forms, which is a contradiction. Therefore, every $g \in C$ can be joined to 1 by a walk that does not cross $\mathrm{Fix}(S)$.

It remains to prove that if $1 \neq s_0 \in S$ then every walk from 1 to s_0 must cross $\mathrm{Fix}(S)$. Let π be any walk from 1 to s_0, and let $(s_1, s_2, \ldots, s_n)$ be the corresponding word representing s_0. Since $1 \in C$ and $s_0 \notin C$, there exists an index k and $1 \neq s \in S$ such that $u := s_1 \cdots s_{k-1} \in C$ and $us_k \in sC$.

Let $(g_1, \ldots, g_n)$ be the normal form for u. Then either $g_1 \in (G_v)^S$ or $g_1 \notin G_v$, whereas the normal form for us_k begins with an element of $s(G_v)^S$ since $us_k \in sC$. In order to affect the first syllable of the normal form, s_k must commute with $g_2, \cdots, g_n$, hence $t_k = g_1 s_k g_1^{-1}$. If $g_1 \notin G_v$, then s_k must commute with g_1 as well. There are three cases to consider.

(1) If $g_1 \notin G_v$, then s_k commutes with u, so $t_k = us_ku^{-1} = s_k$. Therefore, π crosses $\mathrm{Fix}(S)$ while passing from u to us_k.
(2) If $g_1 \in G_v$ and $g_1 \neq s_k^{-1}$, then $(g_1 s_k, g_2, \cdots, g_n)$ is the normal form for us_k. Therefore $g_1 s_k \notin (G_v)^S$, which implies that π crosses $\mathrm{Fix}(S)$ while passing from u to us_k.
(3) If $g_1 = s_k^{-1}$, then $(g_2, \cdots, g_n)$ is the normal form for us_k. But the normal form for us_k must start with an element of G_v, so this case cannot occur.

Therefore, each component of $\mathrm{Cay}(W, \Sigma) \setminus \mathrm{Fix}(R)$ contains exactly one element of S, which implies that S is a hyperreflection. Since S is an arbitrary element of Σ, it follows that (G, Σ) is a hyperreflection system. □

Corollary 11.3. *If G is the internal graph product of $\{G_v\}_{v\in V}$ then $(G, \{G_v\}_{v\in V})$ is a hyperreflection system.*

Proof. Each G_v has a trivial hyperreflection system $(G_v, \{G_v\})$, so the previous theorem implies that $(G, \{G_v\}_{v\in V})$ is a hyperreflection system. □

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